Juggling all these responsibilities be like

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Introduction

The world record for the most balls juggled is 11, by Alex Barron. Juggling is normally thought of as a fringe hobby that is completely unrelated to academics. However, the analyses of juggling patterns can help create never before seen juggling patterns, and they also act as catalysts for discoveries in various fields.

Siteswaps

In order to perform mathematics on juggling, we need a way to represent juggling patterns with numbers. Since juggles are just composed of a ball moving up and down from one hand to the other, all we need is a way to represent the height of a throw and from which hand it leaves. This is exactly what siteswaps do.

Imagine the magnificent Manjul Bhargava, a world renowned magician, musician, and mathematician. Now imagine Professor Bhargava juggling, not from the front view, but from the side. You would see the balls go up and then down, and in some specials patterns the balls would vary in height. It would be difficult to see which hand each ball emerges from, but let's assume Professor Bhargava is a master juggler and, in order to keep his juggles smooth, he swaps hands after each throw. These ideas are all used by siteswap patterns.

A siteswap is a sequence of numbers that specify the amount of time a set of thrown objects are in the air. Each siteswap pattern consists of a sequence of numbers, $t_1 \ldots t_n$, where each t is greater than or equal to 0, and this represent how long the i^{th} object is in the air. An object thrown at time i will land at time t + t_i, the starting time plus the time it takes the ball to land. The times also represent how high each throw is. A ball thrown with the time of 4 seconds needs to be thrown higher in order to land after a ball thrown with the time of 2 seconds.

Let's go back to the majestic scene we had earlier of the side view of Professor Bhargava juggling. From this, we can gather the height of each throw as well as which hand each throw originates from. The only thing we're missing is the time. Siteswaps resolve this by using a timeline starting from 0, with each tick having both a ball thrown and caught. Each throw is represented with an arch curving downwards starting from time t and ending at time t + t_i. Also, a throw of length x means that it will be thrown x ticks after, but in order to be thrown, a ball must first be caught, so a throw of length x will be caught x - 1 ticks after it is thrown.

Since the hands are alternating throws, each throw thrown at time i, where i is an odd number, will be thrown with the same hand, and each throw thrown at time j, where j is an even number, will be thrown by the other hand. Also, if a throw at time i takes t_i seconds to land, then if t_i is odd, the throw will switch hands, and if t_i is even, the throw will land in the same hand. This can be better visualized on the siteswap timeline. A throw thrown with the left hand at time i = 2 with an air time of $t_2 = 3$ will land at time i = 5. Since the even times are thrown with the left hand and odd times with the right, the ball at time i = 5 will be thrown with the right hand.

Let's take a look at the pattern 441. When analyzing and performing the pattern, we assume that it repeats itself, so 441 is short for 441441441... If we start with our right hand, then the first throw will be a throw of length 4 from our right hand to our right hand. The second throw is the same exact throw but one second later and with our left hand. Then we do a throw of length 1 from our right to our left, and the pattern has finished one cycle.

Note 1 *A throw of length 1 is done by handing off a ball from one hand to* *the other. A throw of length 2 is usually done by moving your hand with the ball in it up, then back down in rhythm.*

Since the pattern 441 is of an odd length, we repeat it but starting with our left hand. The length of a pattern is called its **period**, and if a pattern's period is odd, then the first throw will switch hands after each cycle while if the period is even, the same hand will start each cycle.

Validity of Siteswaps

Before, we assumed that each tick on the timeline had exactly one ball being caught and one ball being thrown. Why is this? If we tried to catch two balls at the same time, it would be incredibly difficult unless we were extremely accurate in our throws. Also, throwing two balls at the same time and having each ball go exactly where you want it to go would also be very challenging. We use this assumption for siteswaps to keep the math clean and the juggling smooth, although there is a way to write these pattern with siteswaps.

How do we make sure a pattern doesn't have any collisions? If we look at the pattern 543, all the balls land at time 5, so that's no good. From this we could conclude that as long as $i + t_i$ are distinct for all i, then the pattern is collision-free. However, this doesn't work for 24 or 369. Our theorem only accounts for the first cycle, but collisions can occur during the second or third cycles. To combat this, we can say that as long as i + t_i modulo n, where n is the length of the pattern, are distinct for all i, then the pattern is collision-free.

Theorem 1 *A sequence t_1, ... t_n where all $t_i \geq 0$ is a valid siteswap pattern if all i + t_i are distinct mod n.*

To check if the pattern 441 is collision-free, we need to check if all i + t_i mod n are distinct. For i = 0, 0 + 4 mod 3 = 1. For i = 1, 1 + 4 mod 3 = 2. For i = 2, 2 + 1 mod 3 = 0. Since they are all different, we can conclude that the pattern 441 has no collisions and is a valid swapsite.

Other Aspects of Siteswaps

We now know how to determine whether or not a pattern is juggle-able. The next step is to find the number of balls needed to juggle a pattern. How do we do this? Well, each ball we throw stays in the air a certain amount of time. During that amount of time we cannot use the same ball until it lands. Since a pattern will have a total amount of airtime by all the balls, we can just find this total airtime by adding up all the numbers in a pattern and then divide it by the number of throws we need to do, and that will give us how many balls we will need.

Theorem 2 *A siteswap pattern t_1, ..., t_n will use b balls, where*

$$b = (t_1 + ... + t_n)/n.$$

Another way to look at this is to reverse engineer it. Each ball we use in a pattern is going to be in the air for some time. Since we never hold a ball for more than one count (except for 2, where we move the ball up then down instead of tossing it straight up), the average air time of each ball will be the total airtime divided by the number of throws in a cycle, which is calculated by taking the total airtime, $t_1 + ... + t_n$, and dividing it by the length of the pattern, n.

You may ask what happens when the number of balls is a fraction? How do we juggle 3 and a half balls? Fortunately, this should never happen for valid siteswaps.

Before, we established that for valid siteswaps, each quantity i + t_i will be distinct mod n. That means that the quantities will be exactly 0, 1, ..., n - 1 in some order. This means that if we sum them, we should get (0 + 1 + ... + n - 1) mod n. Therefore,

$$((1 + t_1) + ... + (n + t_n)) = (0 + 1 + ... + n - 1) \bmod n$$

But if we separate the 't's and the numbers on the left side, we get:

$$((1 + t_1) + ... + (n + t_n)) = (t_1 + ... + t_n) + (0 + 1 + ... + n - 1) \bmod n$$

By subtracting the above two, we find that

$$(t_1 + ... + t_n) = 0 \bmod n$$

This implies that $(t_1 + ... + t_n)$, or the total airtime, will be divisible by n. Thus, the average airtime of each throw of a valid siteswap is guaranteed to be a whole number.

As an example, let's take the patterns 441 and 534 and see how many balls each pattern uses. 4 + 4 + 1 is 9, divided by 3 is 3, so 441 should use 3 balls. 5 + 3 + 4 is 12, divided by 3 is 4, so 534 should use 4 balls even though it's the same number of throws per cycle.

Another question an aspiring juggler-mathematician might ask is how many different siteswap patterns exist for a period n and b balls. The answer comes in the compact form of

$$(b+1)^n - b^n.$$

Thus, for siteswap patterns with a period of 3 using 4 balls, there will be 61 valid patterns. However, this number is an overcount because repeating patterns such as 444 have a period of 1 (4) but are still counted as if it had a period of 3.

Applications

Siteswaps have opened new doors in the fields of juggling, mathematics, and science. Some types of juggling can complicate the siteswaps patterns. For example, is some acts, there can be many jugglers passing props from one person to another. Learning about siteswaps is a step towards developing a way to represent this type of juggling. The standard way to represent multiperson siteswap patterns is to use two siteswap lines facing each other.

A throw to oneself is represented with upward or downward arches, and a throw to one's partner is represented as a straight line from one siteswap line to the other.

Another interesting idea is multiplex juggling, where you can catch and throw more than one ball at the same time. Siteswaps have also allowed for new patterns to emerge. Rather than doing a 5 ball cascade, jugglers can now use mathematics to develop different 5 ball patterns such as 4637, a very complicated pattern that involves tossing balls that vary in height and hands. Studying these types of patterns has also allowed for new mathematical advances to take place in topics such as linear algebra, infinite series, Poincaré series, q-nomial Stirling number identities, new Eulerian number identities, and more.

Siteswaps have also helped create juggling state graphs. These graphs use nodes and links to represent all possible juggling permutations for a certain number of balls. It can help jugglers identify and construct new juggling patterns.

Another interesting thought is that if one could slow down time, it would be much easier to juggle complex patterns. While we can't actually manipulate time, we can artificially slow down time for the balls by adjusting the force of gravity. One way to do this would be to juggle on the moon, whose gravitational pull is about 6 times weaker than on Earth. A more practical way would be to throw the balls onto a slanted surface and let them roll down, thereby slowing the falling rate of the balls.

I think the coolest part about siteswaps is how it takes a fairly common and interesting hobby and uses math to analyze the intricacies and possibilities of it. Writing this paper has let me enjoy these afternoons where I can just immerse myself in trying to discover and understand the beautiful mathematical properties behind juggling. For me, the excitement and curiosity I get about the math of juggling is the embodiment of what the discovery and study of math is meant to be.

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